J. Fluid Mech. (1975), *vol.* 68, *part* 4, *pp.* 625–638
Printed in Great Britain

On the continuum theory for the large Reynolds number spherical expansion into a near vacuum

By N. C. FREEMAN, R. S. JOHNSON, S. KUMAR†
School of Mathematics, University of Newcastle upon Tyne, England

AND W. B. BUSH
Department of Applied Mechanics and Engineering Science, University of California, San Diego

(Received 8 July 1974)

The steady, spherically symmetric flow of a compressible gas is considered. The gas is both viscous and heat-conducting. In the limit of very high Reynolds number ($=\alpha^{-1}, \alpha \rightarrow 0$) and correspondingly low pressure at infinity, the structure of the whole flow field is discussed. The five regions that arise by virtue of the limit $\alpha \rightarrow 0$ are briefly considered. Special care is given to the matching across the overlap domains and the first region (close to, but outside, the sonic point) and the fifth (where the pressure adjusts to its ambient value) are carefully examined. It is argued that the application of appropriate matching principles, together with judicious use of numerical solutions, allows an arbitrary pressure and temperature to be assigned to the background gas.

1. Introduction

Recently, considerable interest has been re-awakened in a classical problem in continuum gasdynamics: the three-dimensional, steady, purely radial expansion of a viscous heat-conducting compressible fluid into a vacuum (or near-vacuum). One of the aims of such a study is to find the shortcomings of a theory based on the Navier–Stokes equations – shortcomings which would not be expected to arise in the corresponding problem based on the Boltzmann equation. The flow structure when the Reynolds number is large is well documented (e.g. Sakurai 1958; Freeman 1970). In the classical picture, when the pressure at infinity is finite, the gas expands from sonic conditions along the supersonic branch (Sherman 1964) of a predominantly inviscid solution, its pressure being raised through a shock wave onto the subsonic branch of an essentially inviscid flow. For low pressures, however, the departure from the supersonic inviscid branch is different and the distinction between the shock layer and subsonic flow blurred. The supersonic expansion is followed by an intermediate region where the flow is supersonic and where convection balances the hoop stress (Bush & Rosen 1971; Freeman & Kumar 1972). The region where the velocity passes from supersonic to subsonic values is a very diffuse shock wave with area change (called the 'shock layer'). Unfortunately, from the analytical standpoint, the appropriate

† Present address: Computer Centre, Hatfield Polytechnic, Hatfield, England.

scalings relevant in the shock layer show that the equations in their original (complete) form must be solved.

The solution in the outer reaches of the shock layer may follow one of two possible solutions. Either the pressure approaches zero as $r^{-\frac{4}{3}}$ (where r is the radial distance), in which case there is a vacuum at infinity (Ladyzhenskii 1962), or the pressure tends towards a (small) constant value related to the flow Reynolds number (Freeman 1970). Both Freeman & Kumar (1972) and Bush & Rosen (1971) describe this structure but the emphasis in the two papers is different. Freeman & Kumar (1972, 1973) support their analytical conclusions for the problem with small, but finite, pressure with numerical results for a single value of the downstream temperature (which is appropriate to the case of no heat transfer). Bush & Rosen (1971) seek to obtain an asymptotic solution for the limit of zero pressure downstream and in so doing assert that the temperature at infinity is necessarily determined by *upstream* conditions. Freeman & Kumar (1972, 1973) assume (although their numerical results are for a single temperature) that the downstream temperature can be prescribed, at least for non-zero pressure there. Also, by the nature of their approach, they imply that this is the case for zero pressure downstream. Although it is doubtful whether any physical interpretation can be applied to the Navier–Stokes equations in the zero-pressure limit, it seems plausible that, at least for non-zero pressure, the 'background gas' (in kinetic-theory terms) should determine the downstream temperature.

The main purpose of this paper is to try and resolve the points indicated above. In order to apply appropriate boundary conditions, a complete study of the flow field both upstream and downstream is required. In particular, an analysis of the structure in the neighbourhood of the sonic point must be made. It is shown that it is possible to construct an asymptotic solution (as the Reynolds number approaches infinity) in which both the pressure and temperature† at infinity are given. These boundary data, together with the existence of a sonic radius, suffice to define the complete solution.

The discussion will be based on the (complementary) techniques of asymptotic expansion and numerical integration. The structure of the flow field from the sonic radius to infinity is briefly outlined and the pertinent details relating to (i) the admissible boundary data and (ii) the functional behaviour that can be tested numerically are extracted. A number of numerical results are also discussed.

2. Formulation of the problem

The equations for the purely radial steady flow of a viscous heat-conducting perfect gas may be written in non-dimensional form (Ladyzhenskii 1962) as

$$w'+\frac{1}{\gamma}\left[\left(\frac{\theta}{w}\right)'+\frac{2\theta}{xw}\right]+\alpha\theta^{\omega}\left[\left(w''-\frac{2w}{x^2}\right)+\omega\frac{\theta'}{\theta}\left(w'+\frac{w}{x}\right)\right]=0, \tag{1}$$

$$\theta+\tfrac{1}{2}(\gamma-1)w^2+\alpha\theta^{\omega}\left[\frac{3}{4\sigma}\theta'+(\gamma-1)w\left(w'+\frac{w}{x}\right)\right]=\tfrac{1}{2}(\gamma+1), \tag{2}$$

$$p=x^2\theta/w. \tag{3}$$

† At times it will be more convenient to use the total heat transfer at infinity.

The pressure P, temperature T, velocity u and distance r have been redefined according to

$$p = \frac{r_1^2}{m}\left(\frac{\gamma}{RT_1}\right)^{\frac{1}{2}} P, \quad \theta = \frac{T}{T_1}, \quad w = \frac{u}{(\gamma RT_1)^{\frac{1}{2}}}, \quad x = \frac{r_1}{r}, \quad \text{with} \quad T_1 = \frac{T_0}{\frac{1}{2}(\gamma+1)},$$

where T_0 is the stagnation temperature,† m is the mass flow rate and r_1 is a characteristic length. [r_1 may be the sonic radius r_*, although a more convenient choice will be made later.] The primes denote derivatives with respect to x and we wish to construct a solution for $x_0 \geqslant x \geqslant 0$, where $x_0(\alpha) = r_1/r_*$.‡

Equations (1) and (2) are, respectively, the momentum and energy equations after the pressure P and density ρ have been eliminated by making use of the continuity equation and equation of state for the gas, i.e.

$$\rho u r^2 = m = \text{constant}, \quad P = \rho RT,$$

where R is the gas constant. The viscosity has been assumed proportional to T^ω $(0 < \omega < 1)$ and the Prandtl number σ is constant. The inverse Reynolds number is denoted by α, where

$$\alpha = \tfrac{4}{3}(\mu_1 r_1/m)$$

and μ_1 is the viscosity evaluated at $T = T_1$. Finally, γ (> 1) is the (constant) ratio of the specific heats. Bush & Rosen (1971) introduce two viscosity coefficients in their formulation but as this introduces no new features in the subsequent analysis this additional complication is avoided here.

The problem discussed here is the solution of (1) and (2) in the limit $\alpha \to 0$, with the boundary conditions

$$u = (\gamma RT)^{\frac{1}{2}} \quad \text{at} \quad r = r_*, \tag{4}$$

$$T \to \text{constant}, \quad P \to \text{constant} \quad \text{as} \quad r \to \infty, \tag{5}$$

although, in passing, solutions are given also for $P \to 0$ as $r \to \infty$, but again with arbitrary T (as $r \to \infty$). In some cases, it will be convenient to replace the temperature condition by

$$r^2 T^\omega dT/dr \to \text{constant} \quad \text{as} \quad r \to \infty, \tag{6}$$

which essentially prescribes the total heat transfer at infinity. In non-dimensional variables, (4) becomes

$$w = \theta^{\frac{1}{2}} \quad \text{at} \quad x = x_0(\alpha). \tag{7}$$

For an appropriately small pressure at infinity, (5) and (6) are taken as

$$p \to p_\infty \alpha^n, \quad n \geqslant 2, \quad \text{as} \quad x \to 0, \tag{8}$$

with either

$$\theta \to \theta_\infty \quad \text{as} \quad x \to 0 \tag{9}$$

or

$$\theta^\omega \theta' \to q_\infty \alpha^{-1} \quad \text{as} \quad x \to 0. \tag{10}$$

The heat transfer is seen to be large (as $\alpha \to 0$). p_∞, θ_∞ and q_∞ will be assumed independent of α.

† Defined as that where $u = r^2 u du/dr = r^2 dT/dr = 0$.

‡ Note that, by virtue of the non-dimensionalization, x_0 becomes a free parameter which is to be assigned, e.g. $x_0 = 1$ or $x_0 = x_0(\alpha)$.

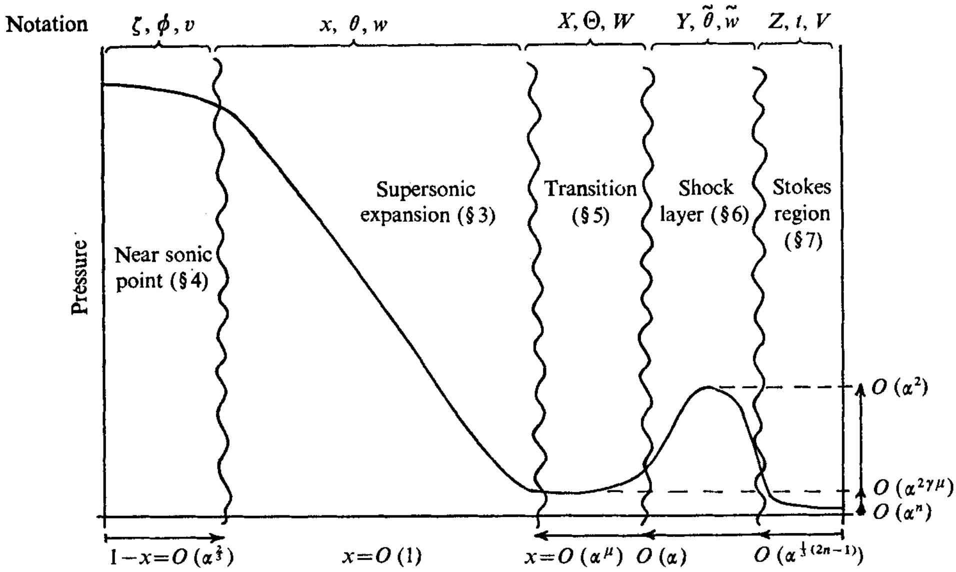

FIGURE 1. Sketch of the pressure giving the order of magnitude of p, the sizes of the various regions and the corresponding notation. Case sketched here is for $n > 2\gamma\mu > 2$.

The analysis is presented for arbitrary p_∞ and θ_∞ (or q_∞) with $n > 2$. The case $n < 2$ corresponds to the classical Laval-nozzle problem, which is extensively described in the literature and will not be discussed here. The solution when $n = 2$ is described by Freeman & Kumar (1972), and for $n > 2$ the results of Freeman & Kumar (1973) are relevant by virtue of their limit $W_0 \to \infty$ (which corresponds here to the limiting process $p_\infty \to 0$ with $n = 2$); however, these studies were only for $\theta = \frac{1}{2}(\gamma+1)$. For $p \sim \alpha^2 p_\infty x^{\frac{2}{3}}$ as $x \to 0$ the reader is referred to Bush & Rosen (1971). Moreover, it is argued here, on the basis of numerical solutions, that θ_∞ may be *freely* chosen and is not determined from upstream as suggested in that paper.

In the next few short sections the various asymptotic regions of the flow field, together with the required matching, are outlined. For convenience, the basic expansion (where $x = O(1)$) is used as the starting point and the regions both upstream and downstream predicted from this. After consideration of the solution in the neighbourhood of the sonic radius, the sections discuss sequentially the solutions as $x \to 0$. The section headings indicate the magnitude of x and reference to figure 1, where the various regions are sketched for one particular range of parameters, may be informative.

3. Basic expansion: $x = O(1)$

The form of the equations suggests that the independent variables may be expanded as

$$\theta = \theta_0 + \alpha\theta_1 + o(\alpha), \quad w = w_0 + \alpha w_1 + o(\alpha), \tag{11}$$

which yields to leading order

$$w_0' + \frac{1}{\gamma}\left[\left(\frac{\theta_0}{w_0}\right)' + \frac{2\theta_0}{x w_0}\right] = 0, \quad \theta_0 + \tfrac{1}{2}(\gamma-1)w_0^2 = \tfrac{1}{2}(\gamma+1). \tag{12a, b}$$

It may be observed that the introduction of terms intermediate between θ_0 and θ_1 or w_0 and w_1 in (11) will produce only perturbations of the inviscid solutions given by (12). In § 4 it is shown that such terms can be eliminated by a suitable rescaling of the variables. The sonic condition is

$$w_0 = \theta^{\frac{1}{2}} \quad \text{at} \quad x = x_0(0)$$

and $x_0(0)$ is set equal to unity, so that r_1 is taken as the inviscid sonic radius (that for $\alpha = 0$) and is presupposed to be the given length scale for the problem. Thus from (12*b*) it follows that

$$w_0 = \theta_0 = 1 \quad \text{at} \quad x = 1, \tag{13}$$

and integration of (12*a*) [with the aid of (12*b*)] gives the entropy equation

$$w_0 \theta_0^{1/(\gamma-1)} = x^2. \tag{14}$$

The solution of (12*b*) and (14) is well known to possess two branches: one subsonic and the other supersonic. Of interest here is the supersonic branch, along which

$$w_0 \to \left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}, \quad \theta_0 \sim \left(\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{2}(\gamma-1)} x^{2(\gamma-1)} \quad \text{as} \quad x \to 0 \tag{15}$$

and
$$w_0 \sim 1 + \frac{2}{(\gamma+1)^{\frac{1}{2}}}\eta^{\frac{1}{2}}, \quad \theta \sim 1 - \frac{2(\gamma-1)}{(\gamma+1)^{\frac{1}{2}}}\eta^{\frac{1}{2}} \quad \text{as} \quad 1 - x \equiv \eta \to 0+. \tag{16}$$

The equations for $w_1(x)$ and $\theta_1(x)$ may be written down and formally solved. However, it is expedient to insert the asymptotic behaviour of w_0 and θ_0 as $x \to 0$ and $x \to 1$ [begun in (15) and (16)] and construct the corresponding behaviour for w_1 and θ_1. This indicates that the expansions (11) are not uniformly valid in both the limits $x \to 0$ and $x \to 1$. In particular, the regions of non-uniformity are obtained as

$$\alpha x^{2\omega(\gamma-1)-1} = O(x^{2(\gamma-1)}) \quad \text{or} \quad x = O(\alpha^{\mu}) \quad \text{as} \quad x \to 0 \tag{17}$$

and
$$\eta^{\frac{1}{2}} = O(\alpha\eta^{-1}) \quad \text{or} \quad \eta = O(\alpha^{\frac{2}{3}}) \quad \text{as} \quad \eta \to 0, \tag{18}$$

where $\mu = [1 + 2(\gamma-1)(1-\omega)]^{-1}$ (< 1). The result (18), which gives the size of the region in the neighbourhood of the sonic point ($x = 1, \alpha \to 0$), is that deduced by Sakurai (1958).

4. Neighbourhood of the sonic point: $1 - x = O(\alpha^{\frac{2}{3}})$

Using the scaling of (18) and the form of the expansions as $\eta \to 0$, the variables are defined as

$$\zeta = \eta\alpha^{-\frac{2}{3}} = (1-x)\alpha^{-\frac{2}{3}}, \quad \text{so that} \quad x_0(\alpha) = 1 - \alpha^{\frac{2}{3}}\zeta_0(\alpha) = 1 + O(\alpha^{\frac{2}{3}});$$
$$\theta \sim 1 + \alpha^{\frac{1}{3}}\phi_0 + \alpha^{\frac{2}{3}}\phi_1; \quad w \sim 1 + \alpha^{\frac{1}{3}}v_0 + \alpha^{\frac{2}{3}}v_1.$$

Now, the leading-order terms from (1) and (2) do not give two independent equations and so going to the next order in ϕ and v (i.e. $O(\alpha^{\frac{2}{3}})$) yields

$$\phi_0 + (\gamma-1)v_0 = 0, \quad av_0' - bv_0^2 + \zeta - \bar{\zeta}_0 = 0, \tag{19\,a, b}$$

where $a = \frac{1}{2}[1+(3/4\sigma)(\gamma-1)]$, $b = \frac{1}{4}(\gamma+1)$ and $\bar{\zeta}_0$ is an arbitrary constant. Equation (19*b*), with $\bar{\zeta}_0 = 0$, is the Ricatti equation given by Sakurai (1958).† The matching condition is

$$v_0 \sim 2(\gamma-1)^{-\frac{1}{2}}\zeta^{\frac{1}{2}} \quad \text{as} \quad \zeta \to \infty, \tag{20}$$

with the boundary condition [from (7)]

$$v_0 = \tfrac{1}{2}\phi_0 \quad \text{at} \quad \zeta = \zeta_0(0), \tag{21}$$

where, as previously noted, $x_0 = 1-\alpha^{\frac{2}{3}}\zeta_0(\alpha)$. Now, (21) and (19*a*) together imply that $v_0 = \phi_0 = 0$ at $\zeta = \zeta_0(0)$.

The solution to (19*b*) may be written in terms of Airy functions as

$$\bar{v}_0 = -(\mathrm{Ai}'(\bar{\zeta})+c\,\mathrm{Bi}'(\bar{\zeta}))/(\mathrm{Ai}(\bar{\zeta})+c\,\mathrm{Bi}(\bar{\zeta})), \tag{22}$$

where $\bar{\zeta} = (b/a^2)^{\frac{1}{3}}(\zeta-\bar{\zeta}_0)$, $\bar{v}_0 = (b^2/a)^{\frac{1}{3}}v_0$ and c is an arbitrary constant. The solution which matches with the supersonic branch [i.e. (20)] is given by $c = 0$. The termination of the solutions (22) (with $c = 0$) due to poles at zeros of $\mathrm{Ai}(\bar{\zeta})$ is of no importance in this analysis. Since the first zero of $\mathrm{Ai}'(\zeta)$ occurs before that of $\mathrm{Ai}(\bar{\zeta})$, in the sense of decreasing $\bar{\zeta}$, the quantity $\zeta_0(0)-\bar{\zeta}_0$ can be chosen to coincide with the first zero of $\mathrm{Ai}'(\bar{\zeta})$.‡ Further, if $\bar{\zeta}_0 \neq 0$, then expansion of (22), as $\zeta\to\infty$, shows that terms $O(\alpha^{\frac{1}{3}})$ must occur in the basic expansion (11). Hence it is convenient to choose $\bar{\zeta}_0 = 0$. There is, correspondingly, a choice of all higher-order terms such that the simplest asymptotic representation may be constructed. Thus, with $\bar{\zeta}_0 = 0$,

$$\zeta_0(0) = -(a^2/b)^{\frac{1}{3}}a_1, \quad \text{where} \quad \mathrm{Ai}'(-a_1) = 0$$

$(-a_1(<0)$ being the first zero), which implies that $\zeta_0(0) < \zeta < \infty$ and thus the position of the viscous sonic radius may be deduced from

$$x_0(\alpha) = 1+\alpha^{\frac{2}{3}}(a^2/b)^{\frac{1}{3}}a_1+\ldots \tag{23}$$

(as a function of r_1 and α).

From the numerical integrations (presented later) it is possible to extract the value of α for a given run. Now, by making use of the next terms in the expansions for $\theta-1$ and $w-1$,

$$\left.\begin{aligned} w &\sim 1+\alpha^{\frac{2}{3}}\frac{a_1}{(\gamma+1)}\left\{2\left(\frac{a^2}{b}\right)^{\frac{1}{3}}-\frac{1}{(ab)^{\frac{1}{3}}}\right\},\\ \theta &\sim 1+\alpha^{\frac{2}{3}}\frac{2a_1}{(\gamma+1)}\left\{2\left(\frac{a^2}{b}\right)^{\frac{1}{3}}-\frac{1}{(ab)^{\frac{1}{3}}}\right\}\end{aligned}\right\} \tag{24}$$

are obtained at the sonic radius $(\zeta = \zeta_0(0))$ and, if numerical accuracy permits, may be used as a check on the solution in the neighbourhood of $r = r_*$.

5. Transition region: $x = O(\alpha^{\mu})$

Returning to the expansion for $x = O(1)$, a non-uniformity is deduced where $x = O(\alpha^{\mu})$ [see (17)] and so

$$x = \alpha^{\mu}X, \quad \theta = \alpha^{\lambda}\Theta, \quad w-\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}} = \alpha^{\lambda}W \tag{25}$$

† The analysis as presented is due to Dr R. E. Grundy (1968, private communication). However, he considered $\zeta_0(0) = 0$. Similar results have been obtained by Collins (1969).

‡ In fact, the first zero of $\mathrm{Ai}'(\bar{\zeta})$ is at $\bar{\zeta} \approx -1{\cdot}02$; the first zero of $\mathrm{Ai}(\bar{\zeta})$ is at $\bar{\zeta} \approx -2{\cdot}34$.

are introduced, where $\lambda = 2(\gamma-1)\mu$. Substituting into (1) and (2), solving for the leading-order terms in Θ and W, and matching with (15) for $X \to \infty$ yields

$$\Theta_0 = \left[\gamma(\gamma+1)(1-\omega)\mu\frac{1}{X}+\left\{\left(\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{2}}X^2\right\}^{(\gamma-1)(1-\omega)}\right]^{1/(1-\omega)} \tag{26}$$

and

$$W_0 = -\frac{\Theta_0}{(\gamma^2-1)^{\frac{1}{2}}}-\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}\frac{\Theta_0^{\omega}}{X} \tag{27}$$

(see Bush & Rosen 1971; Freeman & Kumar 1972). It may be noted in passing that if $\omega = 1$ then $\mu = 1$, and the solution is exponential.

The result (26) affords a particularly useful test that may be exploited in numerical computations. It is found that Θ_0 possesses a minimum such that

$$\Theta_{0\min} = 5{\cdot}5870165 \quad \text{at} \quad X_{\min} = 2{\cdot}1681222 \tag{28}$$

for $\gamma = \frac{5}{3}$ and $\omega = \frac{3}{4}$. Now, since the computations are performed in terms of $\theta = \Theta/\alpha^{\lambda}$ and $Y = x/\alpha = \alpha^{\mu-1}X$ (introduced in § 6) the position of the minimum and the minimum value itself both yield values for α (as $\alpha \to 0$); these may be compared.

The asymptotic expansions begun in (25)–(27) are not uniformly valid as $X \to 0$. Inspection of these solutions immediately places the non-uniformity where

$$\alpha^{\lambda}X^{-1/(1-\omega)} = O(1) \quad \text{or} \quad X = O(\alpha^{1-\mu}) \quad \text{or} \quad x = O(\alpha).$$

6. Shock-layer region: $x = O(\alpha)$

The scalings pertinent here are evidently just that w and θ are both of order unity. Thus for this region the variables are

$$x = \alpha Y, \quad w = \tilde{w}, \quad \theta = \tilde{\theta}, \tag{29}$$

where the tildes denote the (non-dimensional) velocity and temperature written as functions of Y. Now, a cursory glance at (1) and (2) indicates that the application of (29) merely yields the full Navier–Stokes equations in a scaled form. Naively, it might be argued that the worst possible situation has arisen: to proceed the complete (parameterless) problem must be solved. Of course, this is strictly not the case – the boundary (i.e. matching) conditions as $Y \to \infty$ are those from (26) and (27) as $X \to 0$ and not the sonic conditions at $x = x_0$. For reference, the equations are

$$\tilde{w}'+\frac{1}{\gamma}\left[\left(\frac{\tilde{\theta}}{\tilde{w}}\right)'+\frac{2\tilde{\theta}}{Y\tilde{w}}\right]+\tilde{\theta}^{\omega}\left(\tilde{w}''-\frac{2\tilde{w}}{Y^2}\right)+\omega\tilde{\theta}^{\omega-1}\tilde{\theta}'\left(\tilde{w}'+\frac{\tilde{w}}{Y}\right) = 0, \tag{30}$$

$$\tilde{\theta}+\tfrac{1}{2}(\gamma-1)\left(1+\frac{2\tilde{\theta}^{\omega}}{Y}\right)\tilde{w}^2-\tfrac{1}{2}(\gamma+1)+\tilde{\theta}^{\omega}\left[\frac{3}{4\sigma}\tilde{\theta}'+(\gamma-1)\tilde{w}\tilde{w}'\right] = 0, \tag{31}$$

and it is in this form that the problem is integrated numerically (see § 9).

Examination of (26) and (27) shows that in this region the variables must be expanded according to

$$\tilde{\theta} = \tilde{\theta}_0+\alpha^{(1-\mu)/\mu}\tilde{\theta}_1+\dots, \quad \tilde{w} = \tilde{w}_0+\alpha^{(1-\mu)/\mu}\tilde{w}_1+\dots \tag{32}$$

to ensure matching as $Y \to \infty$. The variables $\tilde{\theta}_0$ and $\tilde{w}_0$ satisfy (30) and (31) and, as such, complete (analytical) solutions for these leading-order terms are not available. However, assuming the existence of appropriate solutions, asymptotic solutions of (30) and (31) for $Y \to \infty$ and $Y \to 0$ may be constructed. The procedure in the case $Y \to \infty$ merely confirms that the solutions match with (26) and (27) (for $x \to 0$) (see, for example, Bush & Rosen 1971). In the limit $Y \to 0$ the asymptotic behaviour of $\tilde{\theta}_0$ and $\tilde{w}_0$ can now be presented in detail.

First, it is assumed that $\tilde{\theta}_0$ and $\tilde{w}_0$ possess expansions of the form

$$\tilde{\theta}_0 \sim \tilde{\theta}_\infty + c_1 Y^{\alpha_1}, \quad \tilde{w}_0 \sim d_1 Y^{\beta_1} \quad \text{as} \quad Y \to 0, \tag{33}$$

whence the following values may be deduced for the constants c_1, d_1, α_1 and β_1 (for some $\tilde{\theta}_\infty$).

Case (i). If $\tilde{\theta}_\infty \neq \frac{1}{2}(\gamma+1)$, a solution is

$$\alpha_1 = 1, \quad \beta_1 = 2, \quad c_1 = \tfrac{4}{3}\sigma\tilde{\theta}_\infty^{-\omega}[\tfrac{1}{2}(\gamma+1) - \tilde{\theta}_\infty], \tag{34}$$

whereas if $\tilde{\theta}_\infty = \frac{1}{2}(\gamma+1)$, a solution is

$$\alpha_1 = 4, \quad \beta_1 = 2, \quad c_1 = -\sigma(\gamma-1)\, d_1^2. \tag{35}$$

For either of these two solutions the pressure becomes [from (3) and (8)]

$$p \sim \tilde{\theta}_\infty \alpha^2 Y^2 / d_1 Y^{\beta_1} = \alpha^2 \tilde{\theta}_\infty / d_1, \tag{36}$$

which is constant. This is consistent only if $n = 2$ and $d_1 = \tilde{\theta}_\infty / p_\infty$.

Case (ii). For any $\tilde{\theta}$, a solution is

$$\left.\begin{aligned} \alpha_1 = 1, \quad \beta_1 = \tfrac{1}{2}, \quad d_1 = (2\tilde{\theta}^{1-\omega}/3\gamma)^{\frac{1}{2}}, \\ c_1 = \frac{4\sigma}{3}\left[\tfrac{1}{2}(\gamma+1) - \left(\frac{2\gamma-1}{\gamma}\right)\tilde{\theta}_\infty\right]\tilde{\theta}^{-\omega}. \end{aligned}\right\} \tag{37}$$

The pressure is now

$$p \sim \alpha^2 (\tilde{\theta}_\infty / d_1)\, Y^{\frac{3}{2}}. \tag{38}$$

[It is noted that $\tilde{\theta}_0'(0) = 0$ if $\tilde{\theta} = \gamma(\gamma+1)/2(2\gamma-1)$.]

It is observed that in case (i) the constant d_1 is prescribed in terms of p_∞ and θ_∞. For case (ii), d_1 is completely determined in terms of $\tilde{\theta}_\infty$ alone and the pressure decays like $Y^{\frac{3}{2}}$ as $Y \to 0$ (Ladyzhenskii 1962). Also, in case (i), if $\tilde{\theta}_\infty = \frac{1}{2}(\gamma+1)$ then the complete flow becomes isentropic (as $\alpha \to 0$) since this temperature corresponds to the subsonic branch of (12*b*) and (14). Now, a crucial question is whether solutions exist to, say, the zero-pressure problem of Ladyzhenskii for arbitrary $\tilde{\theta}_\infty$. It is not clear from Ladyzhenskii's original paper whether it was intended that $\tilde{\theta}_\infty$ should be regarded as arbitrarily prescribed or not, but the numerical solutions given later would indicate that this is the case.

The particular interest here is gas entering a very low pressure region where $p \to \alpha^n p_\infty$ ($n \geqslant 2$). It is thus evident that the relevant solution in the shock-layer region (as $Y \to 0$) is case (ii) [(37) and (38)], where the pressure may decrease along the zero-pressure Ladyzhenskii solution until it is as low as $O(\alpha^n)$. This would indicate the limit of validity of the shock-layer solution and occurs when

$$\alpha^2 Y^{\frac{3}{2}} = O(\alpha^n) \quad \text{or} \quad Y = O(\alpha^{\frac{2}{3}(n-2)}), \tag{39}$$

which implies that $x = O(\alpha^{\frac{1}{3}(2n-1)})$, $n > 2$.

7. Pressure-adjustment region: $x = O(\alpha^{\frac{1}{3}(2n-1)})$

From the asymptotic behaviour in (33), and using (37), the scalings given in (39) suggest the new variables Z, V and t, where

$$Y = \alpha^{\delta} Z, \quad \tilde{w} = \alpha^{\frac{1}{2}\delta} V, \quad \tilde{\theta} - \tilde{\theta}_{\infty} = \alpha^{\delta} t \tag{40}$$

and $\delta = \frac{2}{3}(n-2)$ (> 0). [These scalings are equivalent to those used by Freeman & Kumar (1973) when their $W_0^{-\frac{2}{3}}$ is interpreted as α^{δ}.] Substituting (40) into (30) and (31) and retaining the leading-order terms as $\alpha \to 0$ yields

$$V_0'' - \frac{2V_0}{Z^2} = \frac{\tilde{\theta}_{\infty}^{1-\omega}}{\gamma}\left[\frac{V_0'}{V_0^2} - \frac{2}{ZV_0}\right], \tag{41}$$

$$\tilde{\theta}_{\infty} - \tfrac{1}{2}(\gamma+1) + \tilde{\theta}_{\infty}^{\omega}\left[\frac{3}{4\sigma}t_0' + (\gamma-1)V_0V_0'\right] + (\gamma-1)\frac{\tilde{\theta}_{\infty}^{\omega}}{Z}V_0^2 = 0 \tag{42}$$

(where the zero subscripts denote the leading-order terms in V and t). Equation (41) for $V_0(Z)$, which is a balance between viscous and pressure forces, is given in Bush & Rosen (1971) (for $\gamma = 1$). The solution to this equation may be written as

$$\frac{AV_0}{Z} + \ln\left(1 - \frac{AV_0}{Z^2}\right) = B - \frac{\tilde{\theta}_{\infty}^{1-\omega}}{3\gamma}\frac{A^2}{Z^3}, \tag{43}$$

where A and B are arbitrary constants. Now $V_0(Z)$ matches with (33) and (37), as $Z \to \infty$, if $B = 0$. The pressure condition gives

$$p \sim \frac{\tilde{\theta}_{\infty}\alpha^{2(\delta+1)}Z^2}{\alpha^{\frac{1}{2}\delta}(Z^2/A)} = \alpha^n p_{\infty} \quad \text{as} \quad Z \to 0,$$

whence $A = p_{\infty}/\tilde{\theta}_{\infty}$.

Integration of (42) gives the temperature perturbation as

$$t_0 = C - \frac{4\sigma}{3}\left\{\tfrac{1}{2}(\gamma-1)V_0^2 - (\gamma-1)\int_Z^{\infty}\left(\frac{V_0^2}{Z_1} - d_1^2\right)dz_1 + [(\gamma-1)d_1^2 + \bar{t}_0]Z\right\}, \tag{44}$$

where C is an arbitrary constant, d_1 is given in (37) and

$$\bar{t}_0 = [\tilde{\theta}_{\infty} - \tfrac{1}{2}(\gamma+1)]\tilde{\theta}_{\infty}^{-\omega}. \tag{45}$$

It is straightforward to confirm that (44), expanded for $Z \to \infty$, matches exactly with (33) and (37) for any C. In fact, C must match to higher-order terms in the expansion of (30) and (31).

Before examining the implications for the shock-layer region, it is convenient to obtain more information about the expansions here. Setting

$$V = V_0 + \alpha^{\delta}V_1 + o(\alpha^{\delta}), \quad t = t_0 + \alpha^{\delta}t_1 + o(\alpha^{\delta}) \tag{46}$$

and considering the forms taken by $V_1(Z)$ and $t_1(Z)$ as $Z \to \infty$ yields

$$V \sim \left[d_1 Z^{\frac{1}{2}} - \tfrac{1}{3}d_1^2\frac{p_{\infty}}{\tilde{\theta}_{\infty}}Z^{-1}\ldots\right] + \alpha^{\delta}\left[f_1 Z^{\frac{3}{2}} + \frac{C}{3\gamma d_1}\tilde{\theta}_{\infty}^{-\omega}(1-\omega)Z^{\frac{1}{2}}\ldots\right] \tag{47}$$

and

$$t \sim \left[\frac{4\sigma}{3}\bar{\theta}_{\infty}\tilde{\theta}_{\infty}^{1-\omega}Z + C - \frac{4\sigma}{3}(\gamma-1)d_1^3\frac{p_{\infty}}{\tilde{\theta}_{\infty}}Z^{-\frac{1}{2}}\ldots\right]$$

$$+\alpha^{\delta}\left[g_1 Z^2 - \frac{4\sigma}{3}C\tilde{\theta}_{\infty}^{-\omega}\left(\frac{2\gamma-1}{\gamma} + \omega\bar{\theta}_{\infty}\right)Z\ldots\right] \tag{48}$$

as $Z \to \infty$. For simplicity, the following notation has been used:

$$\left.\begin{aligned} \bar{\theta}_\infty &= \left[\tfrac{1}{2}(\gamma+1)-\left(\frac{2\gamma-1}{\gamma}\right)\tilde{\theta}_\infty\right]\tilde{\theta}_\infty^{-1}, \quad f_1 = \tfrac{1}{5}d_1\tilde{\theta}^{-\omega}[\tfrac{1}{2}+\sigma\bar{\theta}_\infty(5-\omega)], \\ g_1 &= -\tfrac{2}{9}\sigma\tilde{\theta}_\infty^{1-2\omega}[4\sigma\bar{\theta}_\infty(\omega\bar{\theta}_\infty+1)+(\gamma+1)\gamma^{-1}\{\tfrac{9}{5}+4\sigma\bar{\theta}_\infty(2+\tfrac{3}{5}\omega)\}]. \end{aligned}\right\} \tag{49}$$

Finally, the temperature as $Z \to 0$ (that is, at infinity) is given by

$$\theta \sim \tilde{\theta}_\infty + \alpha^\delta\left[C+\frac{4\sigma}{3}(\gamma-1)\int_0^\infty\left(\frac{V_0^2}{Z_1}-d_1^2\right)dZ_1\right]. \tag{50}$$

8. Discussion of asymptotic results

From the behaviour of the solution in the pressure-adjustment region it would appear that both the pressure and temperature boundary conditions at $Z = 0$ are necessary to specify the solution. However, it is also essential for the success of this formulation that the requirement for matching with the shock-layer region does not involve further restrictions on the form of the solution. Unfortunately, as analytic solutions to (30) and (31) are not available the behaviour of the various terms as $Y \to 0$ is all that can be used.

The double requirement of matching the shock-layer region both upstream and downstream implies the presence of the appropriate powers of α. To match upstream, terms of the form $\alpha^{m(1-\mu)/\mu}$ [cf. (32)] are necessary; to match to the downstream behaviour terms of the form $\alpha^{q\delta}$ are required. Thus the expansion in the shock-layer region, and hence all other regions, must ultimately include two sequences in α: one generated essentially by the near-field (or sonic-point) behaviour and the other by the far-field behaviour. It is assumed that these two sequences are independent† since it is possible to construct asymptotic solutions for any n (> 2) (i.e. any α^δ). In particular, writing the pressure-adjustment solution [(47) and (48)] in terms of the shock-layer variable Y shows that (32) must be rewritten as

$$\tilde{\theta} \sim \tilde{\theta}_0 + \alpha^{(1-\mu)/\mu}\tilde{\theta}_1 + \ldots + \alpha^\delta\tilde{\tilde{\theta}}_1 + \alpha^{\frac{3}{2}\delta}\tilde{\tilde{\theta}}_2 + \ldots, \tag{51}$$

and similarly for $\tilde{w}(Y; \alpha)$. Now, since (30) and (31) do not involve α, the equations for the pairs of terms $(\tilde{\theta}_1, \tilde{w}_1)$, $(\tilde{\tilde{\theta}}_1, \tilde{\tilde{w}}_1)$ and $(\tilde{\tilde{\theta}}_2, \tilde{\tilde{w}}_2)$ are identical.

By making use of the asymptotic behaviour of $\tilde{\theta}_0$ and $\tilde{w}_0$ as $Y \to 0$, the expansions become

$$\begin{aligned} \tilde{\theta} \sim {} & \left[\tilde{\theta}_\infty + \frac{4\sigma}{3}\bar{\theta}_\infty\tilde{\theta}^{1-\omega}Y + g_1 Y^2 + \ldots\right] + \ldots \\ & + \alpha^\delta\left[h_1 - \frac{4\sigma}{3}\tilde{\theta}_\infty^{-\omega}h_1\left(\frac{2\gamma-1}{\gamma}+\omega\bar{\theta}_\infty\right)Y + \ldots\right] + \alpha^{\frac{3}{2}\delta}[j_1 Y^{-\frac{1}{2}} + \ldots] + \ldots \end{aligned} \tag{52}$$

and

$$\begin{aligned} \tilde{w} \sim {} & [d_1 Y^{\frac{1}{2}} + f_1 Y^{\frac{3}{2}} + \ldots] + \ldots + \alpha^\delta\left[\frac{h_1}{3\gamma d_1}(1-\omega)\tilde{\theta}_\infty^{-\omega}Y^{\frac{1}{2}} + \ldots\right] \\ & + \alpha^{\frac{3}{2}\delta}\left[\frac{j_1}{4\sigma d_1(\gamma-1)}Y^{-1} + \ldots\right] + \ldots, \end{aligned} \tag{53}$$

† Note, however, that terms involving products of the two sequences will necessarily also occur.

where only the terms in $\alpha^{q\delta}$ have been written down. The constants h_1 and j_1 are not determined and $\tilde{\theta}_\infty$, g_1, d_1 and f_1 are defined earlier. The leading-order terms in $(\tilde{\theta}_1, \tilde{w}_1)$, $(\tilde{\tilde{\theta}}_1, \tilde{\tilde{w}}_1)$ and $(\tilde{\tilde{\theta}}_2, \tilde{\tilde{w}}_2)$ may be chosen from three possibilities. Here, the two choices which permit $O(\alpha^\delta)$ and $O(\alpha^{\frac{3}{2}\delta})$ terms to match with (47) and (48) are used. Note, however, that the balance which gives rise to these leading terms (as $Y \to 0$) comes only from the homogeneous equations and since these are linear the undetermined constants h_1 and j_1 are just the arbitrary constants associated with the solution. Thus the constants h_1 and j_1 are chosen such that matching is fulfilled. This leads directly to

$$h_1 = C, \quad j_1 = -\tfrac{4}{3}\sigma(\gamma-1)\, d_1^3 (p_\infty/\tilde{\theta}_\infty). \tag{54}$$

The problem, in terms of matched asymptotic expansions, has now been outlined and it remains only to impose the boundary conditions at infinity ($Z = 0$). In fact, the pressure condition ($p \sim \alpha^n p_\infty$) has already been incorporated in (43) by the appropriate choice of A. For the application of the thermal condition equation (50) is employed. If the temperature is given [see (9)] then it is required that

$$\tilde{\theta}_\infty = \theta_\infty, \quad C = -\frac{4\sigma}{3}(\gamma-1)\int_0^\infty \left(\frac{V_0^2}{Z_1} - d_1^2\right) dZ_1 \tag{55}$$

and the temperature reached in the shock-layer solution using the zero-pressure Ladyzhenskii (1962) expansion is just the arbitrary value at infinity. If, on the other hand, the heat transfer is given, then

$$\tilde{\theta}_\infty^{\omega}(-(4\sigma/3)\bar{t}_0) = q_\infty \quad \text{or} \quad \tilde{\theta}_\infty = \tfrac{1}{2}(\gamma+1) - (3/4\sigma)\, q_\infty, \tag{56}$$

whence, for zero heat transfer, $\tilde{\theta}_\infty = \frac{1}{2}(\gamma+1)$.

9. Numerical solutions and conclusions

It has already been pointed out that, in the shock-layer region, it is necessary to obtain a solution of the complete equations of motion (30) and (31) to define the structure in that particular region. This is, however, simpler than finding a solution to the complete problem since only matching conditions must be satisfied. Since, to first order as $\alpha \to 0$, the Ladyzhenskii solution [(33) with (37)] gives a complete solution which is independent of the pressure at infinity, this parameter is eliminated from this calculation. However, it is obvious from the Ladyzhenskii expansion (33) that the asymptotic downstream behaviour depends critically on the assumed value of $\tilde{\theta}_\infty$. Now, it is only possible to examine the asymptotic solutions upstream and downstream in the shock-layer region and thus it is not apparent whether $\tilde{\theta}_\infty$ is a consequence of the upstream matching condition or not. Although it could be argued from physical considerations that it might be determined from the temperature of the background gas, the limitations of the Navier–Stokes equations themselves (for small pressures) make this assertion highly questionable. The only practical way of deciding whether $\tilde{\theta}_\infty$ is indeed freely available is to construct numerical solutions starting with the Ladyzhenskii expansion (for a given value of $\tilde{\theta}_\infty$) and to observe if upstream conditions can be satisfied.

Numerical solution of the Navier–Stokes equations for spherically symmetric flow has been carried out by Gusev & Zhbakova (1969) and Rebrov &

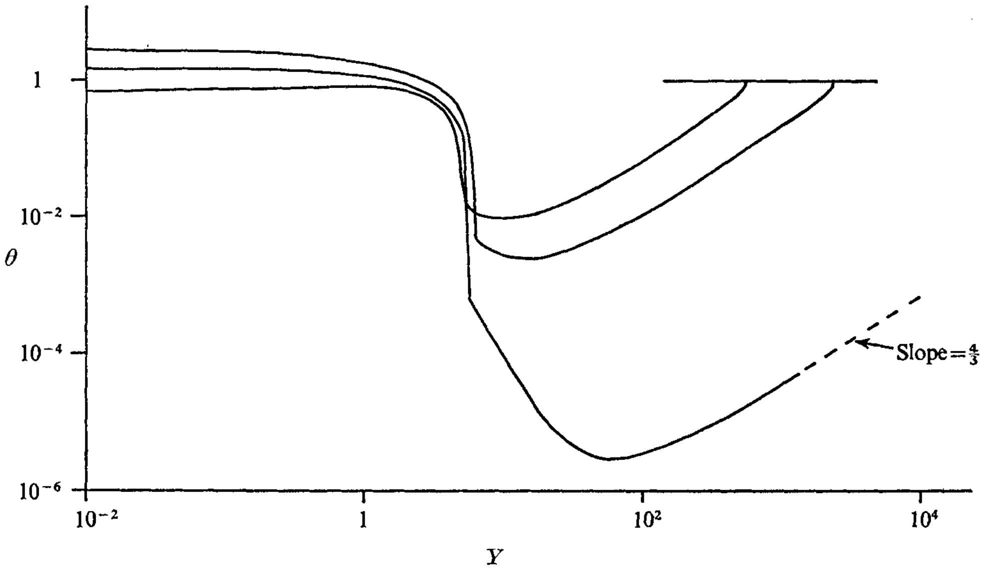

FIGURE 2. Numerical solutions obtained for $\tilde{\theta}_\infty = \frac{8}{3}$, $\alpha = 0{\cdot}000434$; $\tilde{\theta}_\infty = \frac{22}{15}$, $\alpha = 0{\cdot}000000535$ and $\tilde{\theta}_\infty = \frac{2}{3}$, $\alpha = 0{\cdot}00175$. ---, extension based on the theoretical behaviour.

Chekmaryov (1970). Owing to specific assumptions about the viscosity dependence these results are not suitable for comparison with the present theory but their techniques were developed by Freeman & Kumar (1972), who made use of the asymptotic expansion appropriate to a small (but non-zero) pressure downstream. In that analysis it was found that solutions could be constructed only if very accurate starting values were obtained by using a large number of terms in the asymptotic expansions. A similar approach was tried with the Ladyzhenskii expansion but this failed owing to the appearance of a logarithmic term in the velocity expansion. This term, which behaves like $Y^{\frac{7}{2}} \ln Y$ as $Y \to 0$, is associated with the solution for finite pressure and corresponds to the existence of two possible types of solution.† Such a behaviour then requires that the coefficient of $Y^{\frac{7}{2}}$ be arbitrary and this, together with the logarithmic term, produces an extremely complicated sequence. Consequently, this appears to rule out any attempt to construct many terms in the expansion even if, as in the previous exercise, the computer were programmed to do this. However, it had been noticed that the procedure of calculating a large number of terms in the asymptotic expansion only served to give an even closer relationship between the values of the derivatives calculated from the equations and those found by using the asymptotic expansion. This suggested that it might be feasible to obtain a satisfactory solution by a 'trial and error' choice of the coefficient of $Y^{\frac{7}{2}}$. This proved

† This type of behaviour is, of course, very familiar in second-order linear differential equations when the expansion of one solution of the homogeneous equation has a first term corresponding to a term in the expansion of the other. It is, however, surprising that a similar result appears in this highly nonlinear problem, where no obvious relation exists between the two types of solution.

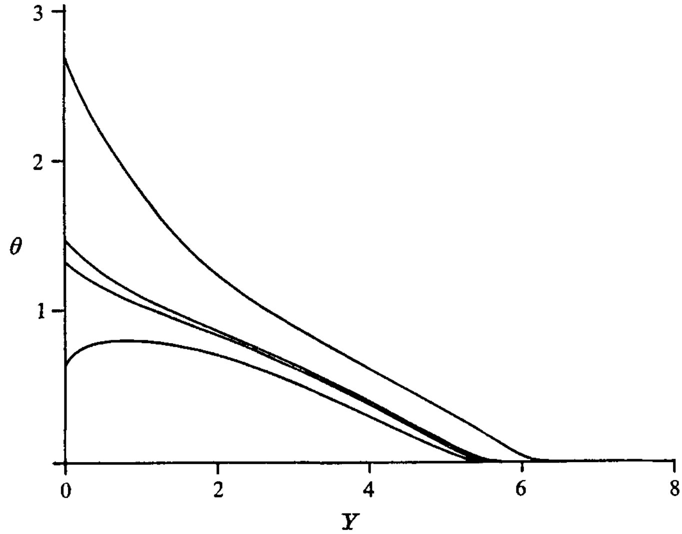

FIGURE 3. Temperature distribution near infinity ($Y \to 0$) from numerical solutions for $\tilde{\theta}_\infty = \frac{8}{3}$, $\alpha = 0{\cdot}000434$; $\tilde{\theta}_\infty = \frac{22}{15}$, $\alpha = 0{\cdot}000000535$; $\tilde{\theta}_\infty = \frac{4}{3}$, $\alpha = 0{\cdot}000125$ and $\tilde{\theta}_\infty = \frac{2}{3}$, $\alpha = 0{\cdot}00175$.

possible although it was necessary to obtain very accurate values of this coefficient (e.g. to within 1 part in 10^8) if the integrations were to proceed successfully (and not lead to physically unrealistic situations).

The solutions obtained for different values of $\tilde{\theta}_\infty$ are shown in figure 2. Here the temperature is plotted against the scaled variable Y on a log–log plot. Since the integrations start at infinity, the inverse Reynolds number α is not known *a priori*. It may, however, be computed from the minimum value of the temperature [from (28)] or the value of Y at which sonic conditions are achieved. In fact, these values agree to within a fraction of a per cent for all the solutions! Unfortunately, the values of α for the various solutions are widely different. These values can be adjusted by judicious but tedious variation of the starting conditions. A similar procedure was adopted in Freeman & Kumar (1972) and the results are shown in their figure 3. In view of the large amount of computer time used in these calculations it was not thought worthwhile to attempt an adjustment of α in the present work. As a number of solutions has been obtained for a wide range of downstream temperatures (even if the values of α are not comparable), there seems to be little doubt that solutions can be obtained for any value of this temperature. Consequently it is possible to match to any downstream pressure $O(\alpha^2)$, or smaller, through the pressure-adjustment region.

The various regions described earlier can easily be identified in figure 2: from the downstream end these are the shock layer, the transition region, inviscid region and finally the sonic transition region. Also, the temperatures are plotted in figure 3 on a linear basis. It is evident that various starting temperatures $\tilde{\theta}_\infty$ at $Y = 0$ lead to similar results: in particular the approach to the shock-layer region (where θ is very small) depends only weakly on either $\tilde{\theta}_\infty$ or α.

Since detailed results were available near the sonic point it was possible to

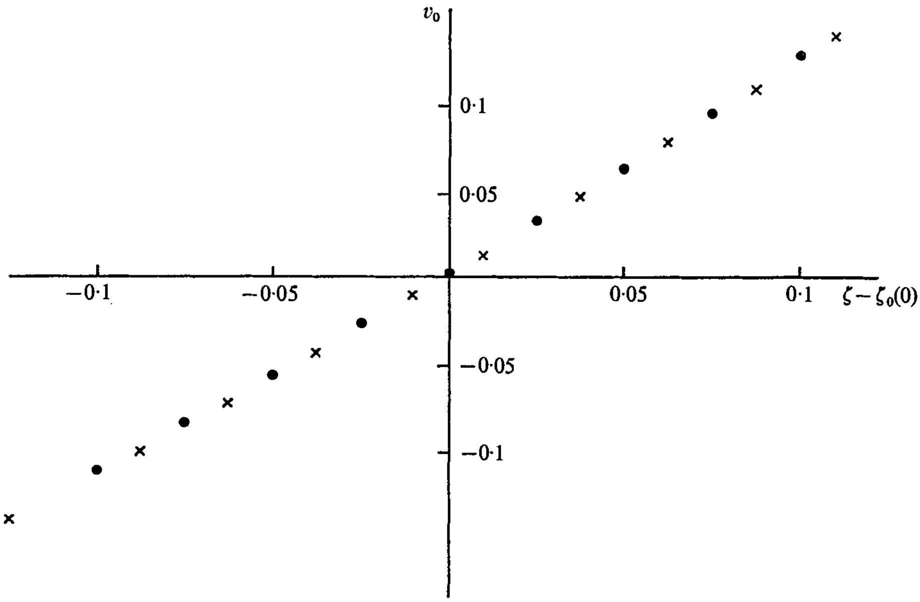

FIGURE 4. Behaviour near the sonic point from numerical solutions. ●, $\theta_\infty = \frac{2}{3}$, $\alpha = 0{\cdot}00175$; ×, $\theta_\infty = \frac{8}{3}$, $\alpha = 0{\cdot}000434$. For definition of the variables see §4.

verify the behaviour predicted by (22) and (24) in this region (as shown in figure 4). It is seen that the appropriate scaling, for different values of α, enables the solutions to collapse onto the same curve.

In conclusion, this paper shows how the flow can expand from sonic conditions, pass through a supersonic–subsonic transition and eventually reach an arbitrary (low) pressure and, apparently, arbitrarily prescribed temperature at infinity. Of course, this has been made possible only by permitting the Reynolds number to be very large and by an extensive use of numerical integrations. With these restrictions, it is reasonably certain that the behaviour of the system of differential equations predicted above is correct.

REFERENCES

BUSH, W. B. & ROSEN, R. 1971 *SIAM J. Appl. Math.* **21**, 393–406.

COLLINS, R. L. 1969 Ph.D. thesis, University of Southern California.

FREEMAN, N. C. 1970 Continuum and non-continuum theories of the steady spherically symmetric expansion into a near vacuum. In *Proc. 7th Int. Symp. on Rarefied Gas Dynamics, Pisa, Italy* (to appear).

FREEMAN, N. C. & KUMAR, S. 1972 *J. Fluid Mech.* **56**, 523–532.

FREEMAN, N. C. & KUMAR, S. 1973 *J. Fluid Mech.* **59**, 391–396.

GUSEV, V. N. & ZHBAKOVA, A. V. 1969 *Adv. in Appl. Mech. Suppl.* **5**, 847–862.

LADYZHENSKII, M. D. 1962 *J. Appl. Math. Mech.* **26**, 965–974.

REBROV, A. K. & CHEKMARYOV, S. F. 1970 Spherical expansion of the viscous heat conducting low density gas into a flooded space. In *Proc. 7th Int. Symp. on Rarefied Gas Dynamics, Pisa, Italy* (to appear).

SAKURAI, A. 1958 *Quart. J. Mech. Appl. Math.* **11**, 274–289.

SHERMAN, F. S. 1964 *Arch. Mech. Stos.* **2**, 471–490.